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Holonomic Quantum Computation with Josephson Networks

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It is well known that it is possible to build quantum gates based entirely on the action of geometric phases. This approach is called holonomic quantum computation. Here we show that it is possible to generate and detect non-Abelian geometric phases in the charge dynamics of a network of mesoscopic Josephson junctions. The scheme can be used to realize a universal set of quantum gates.

1. Introduction The most common paradigm how to perform a quantum computation is to prepare the quantum register in some initial state and then to apply a well-defined sequence of one-qubit and two-qubit (or, in general, N-qubit) operations. These unitary operations – the so-called quantum gates – transform the quantum state according to the given quantum algorithm. Finally, the register is measured in order to read out the result of the computation.

An implementation of a quantum computer has to choose a physical system to represent the two-state system and to define the dynamical variable which can be controlled and measured in the desired way. In particular it has to specify suitable Hamiltonians and to devise ways to realize the quantum gates.

An obvious way to implement unitary operations is to utilize the time evolution of a system (see for example Ref. [1]). Initially the considered qubits are prepared in an "idle state", typically the eigenstate of a starting Hamiltonian. At the beginning of the operation, an additional term in the Hamiltonian is switched on and acts for a well-specified time (the operation time τ_{op}). Afterwards the Hamiltonian is switched back to the initial one. Thus, the gate operation is realized by the (free or driven) time evolution of a more or less complex physical system. The characteristic scale for τ_{op} is given by some intrinsic time scale of the system dynamics.

There is an alternative approach to generate quantum gates by making use of geometric phases. It is well known that in quantum mechanics geometric phases are associated with the cyclic evolution of quantum states [2, 3]. It has been demonstrated recently that by combining geometric and dynamical phases it is possible to obtain a universal set of quantum gates [4–6].

Interestingly, it has been pointed out by Zanardi and co-workers that, in fact, quantum computation can be implemented without using dynamical phases at all [7]. In this

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novel approach to quantum computation, information is encoded in a degenerate eigenspace of a parametric family of Hamiltonians. The gates are realized in terms of the non-Abelian Berry connection obtained by changing the control parameters adiabatically along closed contours in the parameter space [8]. If the connection is irreducible a universal set of unitary transformations can be realized by geometric means only.

In order to contrast this way of doing gate operations with the standard "dynamical approach" the term *holonomic quantum computation* has been coined. Until now only one scheme has been proposed to realize holonomic quantum computation with trapped ions [9]. On the other hand, it would be particularly interesting to have an implementation for a solid-state system. Quantum bits realized in superconducting and semiconducting systems appear to be promising candidates for a future technology due to their scalability and integrability. In particular the impressing experimental progress in the development of quantum coherent superconducting devices [10–14] stimulates a more detailed investigation of geometric phase effects in such systems.

Here we show that non-Abelian phases can appear in the quantum dynamics of Josephson networks. As it will become clear in the following non-Abelian phases can be detected by a measurement of the adiabatic charge dynamics of the nanocircuit. The idea is to construct a circuit of mesoscopic superconducting islands and Josephson junctions with a low-energy Hamiltonian analogous to the one in Refs. [9, 15] for the one-qubit setup. Finally, we demonstrate that two of these qubits can be coupled in order to implement a controlled phase shift gate.

2. Abelian and Non-Abelian Holonomies Holonomies can be either simple Abelian phase factors (Berry phases) or they can correspond to non-Abelian operations. In this section we briefly sketch how holonomies are obtained by cyclic parameter evolution [7, 8].

Consider an eigenstate of a quantum system (degenerate or non-degenerate) which depends on the external parameters $\lambda = \lambda_1, \ldots, \lambda_N$. A cyclic adiabatic evolution of λ brings the system back to its initial state (for non-degenerate states) or keeps it within the degenerate subspace (for degenerate initial states). The final state $|\psi(\tau_{\rm op})\rangle$ is related to the initial state $|\psi(0)\rangle$ by

$$|\psi(\tau_{\rm op})\rangle = U\ |\psi(0)\rangle \exp\left[-\frac{i}{\hbar}\int\limits_0^{\tau_{\rm op}} E(t)\ {\rm d}t\right].$$

Here U denotes a unitary operator which depends only on the path in the parameter space, and the exponential factor accounts for the dynamical phase acquired during the evolution (the momentary eigenvalue of the energy is denoted by $E(t)$). The mapping U can be evaluated by introducing a set of basis states $|n(\lambda)\rangle$, $n = 1, \ldots, N$ in the N-fold degenerate subspace (the basis is local at each point of the parameter space). Then the matrix elements of U are obtained by considering the evolution of these basis states

$$|n(\lambda(t))\rangle = \sum_m U_{nm}(t)\ |m(\lambda(t))\rangle\,.$$

For a cyclic evolution in the parameter space one finds

$$U(\tau_{\rm op}) = \mathbf{P} \exp\left[\int\limits_0^{\tau_{\rm op}} A_\lambda\ {\rm d}\lambda\right], \tag{1}$$

where **P** denotes the path-ordering symbol and the connection forms are given by

$$A_{nm,\lambda} = \langle n| \frac{\partial}{\partial\lambda} |m\rangle \, .$$

In the case $N = 1$ (non-degenerate initial state) the path ordering symbol can be dropped and one obtains the expression for the scalar Berry phase.

3. Design of a Suitable Josephson Junction Circuit Our focus is to realize the Hamiltonian used in Refs. [9, 15] in nanofabricated superconducting circuits. This is indeed possible due to the great flexibility in design offered by Josephson circuits. The building block to generate non-Abelian phases is the network shown in Fig. 1. It consists of three superconducting islands (labeled by $j = 0, \mathrm{a}, 1$ in analogy to Ref. [9]) connected to a fourth one (labeled with 'e') by tunable Josephson couplings. A gate voltage is applied to each island via a gate capacitance. The device operates in the charging regime, that is the Josephson energies J_0, J_a, J_1 are much smaller than the charging energies E_c of an indivual island and $\tilde{E}_\mathrm{c}$ of the entire setup. For temperatures $T \ll J_j \ll \tilde{E}_\mathrm{c} < E_\mathrm{c} < \Delta$ only the two charge states corresponding to the presence of *zero* or *one* excess Cooper pair on each island become important (Δ denotes the superconducting gap). We ignore quasiparticle tunneling since it is suppressed at low temperature. The gate charges $n_{xj} = C_\mathrm{g} V_{xj}/2e$ on each island can easily be tuned by changing the gate voltages. The Josephson couplings can be controlled e.g. if the junction is designed as a Josephson interferometer (a loop with two junctions in parallel and pierced by a magnetic field). Conditions are arranged such that the number of charges is fixed with a total of one excess Cooper pair in the four-island setup. There are four charge states $|j\rangle$ of the system corresponding to the position of the excess pair on island j.

By choosing the gate charges appropriately around $n_{xj} = 1/2$, $|j\rangle$ is the state with minimum charging energy ($j = 0, \mathrm{a}, 1$). If the system is in the state $|\mathrm{e}\rangle$ there is an extra charging energy δE_c.

Thus, we obtain the Hamiltonian on complete analogy of Refs. [9, 15]

$$H = \delta E_\mathrm{c} \, |\mathrm{e}\rangle \langle \mathrm{e}| + \tfrac{1}{2} \left[J_0 \, \mathrm{e}^{-i\alpha_0} \, |\mathrm{e}\rangle \langle 0| + J_\mathrm{a} \, \mathrm{e}^{-i\alpha_\mathrm{a}} \, |\mathrm{e}\rangle \langle \mathrm{a}| + J_1 \, \mathrm{e}^{-i\alpha_1} \, |\mathrm{e}\rangle \langle 1| + \mathrm{h.c.} \right] , \quad (2)$$

where

$$J_j = J_j(\bar{\Phi}_j) = \sqrt{(J_{j\mathrm{r}} - J_{j\mathrm{l}})^2 + \, 4 J_{j\mathrm{r}} J_{j\mathrm{l}} \cos^2(\pi \bar{\Phi}_j)} \, , \quad (3)$$

$$\tan(\alpha_j) = \frac{J_{j\mathrm{r}} - J_{j\mathrm{l}}}{J_{j\mathrm{r}} + J_{j\mathrm{l}}} \, \tan(\pi \bar{\Phi}_j) \, , \quad (4)$$

here $\bar{\Phi}_j = \Phi_j/\Phi_0$ ($j = 0, \mathrm{a}, 1$) are the external magnetic fluxes in units of the flux quantum $\Phi_0 = h/2e$ and $J_{j\mathrm{l}}$, $J_{j\mathrm{r}}$ denote the Josephson energies of the left and right junction

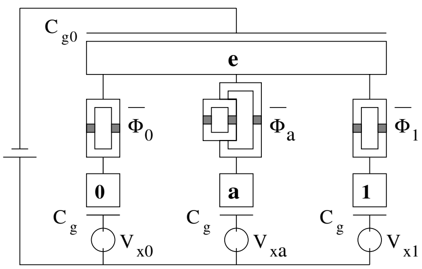

Fig. 1. Josephson junction circuit for the generation of non-Abelian geometric phases. The Josephson energy of an asymmetric SQUID loop cannot be switched off completely in accordance with Eq. (3). Since $J_\mathrm{a} = 0$ may be desirable for quantum computation the loop is designed such that this condition can be satisfied

of each SQUID loop in Fig. 1, respectively. In the following we will refer to the fluxes $\{\bar{\Phi}_0, \bar{\Phi}_a, \bar{\Phi}_1\}$ as *control* parameters and to their manifold $\mathcal{M}$ as the *control manifold*. In general, all the SQUID loops could be asymmetric, although it is not necessary for the purpose of our discussion.

The Hamiltonian defined in Eq. (5) has the following two degenerate eigenstates (not normalized) with zero-energy eigenvalue:

$$|D_1\rangle = -J_a \,e^{-i\alpha_a} |0\rangle + J_0 \,e^{-i\alpha_0} |a\rangle ,$$

$$|D_2\rangle = -J_1 \,e^{-i\alpha_1} (J_0 \,e^{i\alpha_0} |0\rangle + J_a \,e^{i\alpha_a} |a\rangle) + (J_0^2 + J_a^2) |1\rangle . \quad (5)$$

We stress that the charge degree of freedom is at the core of our analysis. By proper manipulation of the coupling energies J_j (including their phases) it is possible to realize adiabatic charge transfer between the islands as can be concluded directly from Eq. (5). For the purpose of quantum computation, the charge states will be used to encode the information.

4. One-qubit Operations Proceeding along the lines of Ref. [9], we point out the necessary ingredients and the differences which arise in the case of the Josephson junction setup.

We identify the states $|0\rangle$ and $|1\rangle$ with the computational basis of the qubit. For idle periods (i.e. between operation periods of the qubit) the couplings J_0, J_1 are switched off. Since the implementation guarantees that the excess Cooper pair during idle periods is shared only between the states of the computational basis (i.e. it will never be found on the 'e' or the 'a' island), the pair of the islands '0' and '1' looks like a common Josephson charge qubit from the outside [16].

For the implementation, it is sufficient to provide explicit representations for the gates $U_1 = e^{i\Sigma_1 |1\rangle\langle 1|}$ and $U_2 = e^{i\Sigma_2 \sigma_y}$, describing rotations of the qubit state about the z axis and the y axis, respectively. We note that only one asymmetric SQUID loop (as shown in Fig. 1) is required to implement the one-qubit operations.

The gate U_1 is a phase shift for the state $|1\rangle$ while the state $|0\rangle$ remains decoupled, i.e. $J_0 \equiv 0$ during the operation. In the initial state we have $J_1 = 0$, so the eigenstates $\{|D_1\rangle, |D_2\rangle\}$ correspond to the logical states $\{|0\rangle, |1\rangle\}$.

The control parameters $\bar{\Phi}_a, \bar{\Phi}_1$ evolve adiabatically along the closed loop C_1 in the $(\bar{\Phi}_a, \bar{\Phi}_1)$-plane from $\bar{\Phi}_1 = 1/2$ to $\bar{\Phi}_1 = 1/2$. By using the formula for holonomies one can show that this cyclic evolution produces the gate U_1 with the phase Σ_1,

$$\Sigma_1 = \sigma_1 \oint_{\mathcal{S}(C_1)} d\bar{\Phi}_a \, d\bar{\Phi}_1 \frac{\sin(2\pi\bar{\Phi}_1)}{(J_1^2(\bar{\Phi}_1) + J_a^2(\bar{\Phi}_a))^2} , \quad (6)$$

where $\mathcal{S}(C_1)$ denotes the surface enclosed by the loop C_1 in $\mathcal{M}$ and $\sigma_1 = 4\pi^2 J_1^2(0)(J_{al}^2 - J_{ar}^2)$.

Similarly we can consider a closed loop C_2 in the $(\bar{\Phi}_0, \bar{\Phi}_1)$-plane at fixed $\bar{\Phi}_a = 0$, and let the control parameters $\bar{\Phi}_0$ and $\bar{\Phi}_1$ undergo a cyclic adiabatic evolution with starting and ending point $\bar{\Phi}_0 = \bar{\Phi}_1 = 1/2$ (see Fig. 2). This operation yields the gate U_2 with the phase Σ_2

$$\Sigma_2 = \sigma_2 \oint_{\mathcal{S}(C_2)} d\bar{\Phi}_1 \, d\bar{\Phi}_0 \frac{\sin(\pi\bar{\Phi}_0)\sin(\pi\bar{\Phi}_1)}{(J_a^2(0) + J^2(\bar{\Phi}_1) + J^2(\bar{\Phi}_0))^{3/2}} , \quad (7)$$

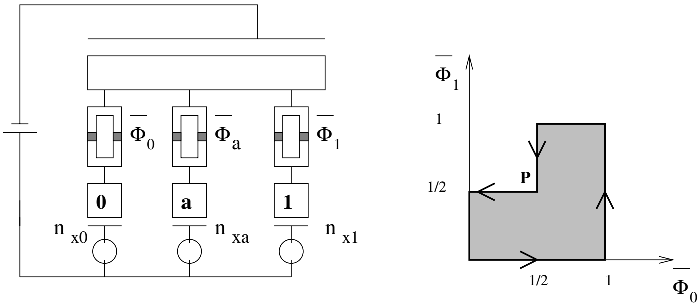

Fig. 2. Starting from $P = (1/2, 1/2)$ and adiabatically following the drawn path, the gate $U = e^{i\frac{\pi}{2}\sigma_y}$ can be achieved

where $\mathcal{S}(C_2)$ denotes the surface enclosed by the loop C_2 in $\mathcal{M}$, and $\sigma_2 = 4\pi^2 J^2 (J_{\rm al} + J_{\rm ar})$ where we have assumed $J_0(0) = J_1(0) =: J$.

5. Two-qubit Coupling and Two-bit Controlled Phase Shift In order to obtain a universal set of operations we need to provide a two-qubit gate. It turns out that it is possible to implement a conditional phase shift $U_3 = e^{i\Sigma_3 |11\rangle\langle 11|}$ by coupling the qubits via Josephson junctions. These junctions should be realized as symmetric SQUID loops such that the coupling can be switched off. The capacitive coupling due to these SQUID loops can be neglected if the capacitances of the junction are sufficiently small [17].

The implementation of the conditional phase shift U_3 is analogous to the single-qubit phase shift U_1. That is, we need to find a subspace of the 16-dimensional space of two-qubit states $\{|00\rangle, |01\rangle, \ldots, |ee\rangle\}$ for which we can construct a Hamiltonian with the same structure as in Eq. (2). This subspace is given by the states $\{|00\rangle, |11\rangle, |ea\rangle, |ee\rangle\}$. Note that now we have to set $\delta E_c = 0$ while this was not necessary in the one-qubit case. By coupling the 'e' island of each qubit with the '1' island of the other (as shown in Fig. 3) and by switching on the coupling $J_a^{(2)}(\bar{\Phi}_a^{(2)})$ of the second qubit we obtain the Hamiltonian

$$H_{\rm 2qubit} = \frac{1}{2}\,[J_a^{(2)}\, e^{-i\alpha_a^{(2)}}\, |ee\rangle\,\langle ea| + J_X\,|ee\rangle\,\langle 11| + {\rm h.c.}]\,. \tag{8}$$

The matrix element $J_X = J_X(\bar{\Phi}_4, \bar{\Phi}_5)$ is given by

$$J_X = -(1/2)\, J_{\rm e1}(\bar{\Phi}_4)\, J_{\rm 1e}(\bar{\Phi}_5)\, \mu\,, \tag{9}$$

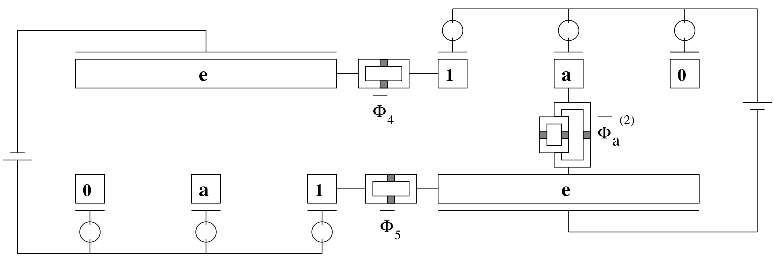

Fig. 3. Coupling between two qubit setups for the implementation of the gate U_3

where $\mu = [1/\delta\tilde{E}_c^{+-} + 1/\delta\tilde{E}_c^{-+}]$. Here $\delta\tilde{E}_c^{+-}$ and $\delta\tilde{E}_c^{-+}$ denote the charging energy difference between the initial and the intermediate state (see below). The coupling is of second order since the inter-qubit coupling junctions change the total number of pairs on each one-bit setup. Thus the coupling occurs via intermediate charge states which lie outside the Hilbert space of the two-qubit system. These are states, e.g. without excess Cooper pair on the first qubit and two excess pairs on the second qubit. In Eq. (9) we have abbreviated the charging energy difference between the corresponding state and the initial qubit state by $\delta\tilde{E}_c^{-+}$, and we have denoted the external magnetic fluxes in the coupling SQUID loops by Φ_4 and Φ_5.

While $J_X(\bar{\Phi}_4, \bar{\Phi}_5)$ is the only off-diagonal coupling of second order, there are also second-order corrections of the diagonal elements, i.e. of the energies of the two-qubit states. These corrections would lift the degeneracy and thus would hamper the geometric operation which is based on the degeneracy of all states. It is therefore crucial that it is possible to compensate these corrections and to guarantee the degeneracy. It is easy to see that by adjusting the gate voltages the energy shifts can be canceled. Note that during the geometric operation the values of the Josephson couplings are changing and therefore also the energy shifts are not constant. Consequently their compensation by means of the gate voltages has to follow the evolution of the parameters.

Let us now show explicitly how the gate U_3 can be achieved. To this aim, we consider a closed loop C_3 in the $(\bar{\Phi}_a^{(2)}, \bar{\Phi}_4)$-plane at fixed $\bar{\Phi}_5 = 0$. If the control parameters $\bar{\Phi}_4$ and $\bar{\Phi}_a^{(2)}$ undergo a cyclic adiabatic evolution with starting and ending point $\bar{\Phi}_4 = 1/2$, $\bar{\Phi}_a^{(2)} = 0$, the geometric phase obtained with this loop is

$$\Sigma_3 = \sigma_3 \oint_{S(C_3)} \mathrm{d}\bar{\Phi}_a^{(2)}\, \mathrm{d}\bar{\Phi}_4 \, \frac{\mu^2 \sin(2\pi\bar{\Phi}_4)}{\left((J_a^{(2)}(\bar{\Phi}_a^{(2)}))^2 + \mu^2 J^2 J_{e1}^2(\bar{\Phi}_4)\right)^2} ,$$

with $\sigma_3 = 4\pi^2 J^4 (J_{al}^2 - J_{ar}^2)$ and $J_{e1}(0) = J_{1e}(0) =: J$. As we have mentioned in the introduction, some caution is required before regarding this scheme ready for implementation. In practice it will be difficult to achieve perfect degeneracy of all states. Thus the question is imposed to which extend incomplete degeneracy of the qubit states is permissible. Clearly, the adiabatic condition requires the inverse operation time τ_{op} to be smaller than the minimum energy difference to the neighboring states $\tau_{op}^{-1} \ll \min\, \delta\tilde{E}_c, J_j, J_X$. On the other hand, if the degeneracy is not complete and the deviation is on the order ϵ one can show by modifying the derivation of Eq. (4) in Ref. [7] that for $\epsilon \ll \tau_{op}^{-1}$ the holonomies can be realized to a sufficient accuracy. This inequality expresses the requirement that the operation time be still small enough in order to not resolve small level spacings of the order ϵ.

There is another important constraint on τ_{op}. As the degenerate states in Eq. (5) are different from the ground state of the system, τ_{op} must not be too large in order to prevent inelastic relaxation. The main origin for such relaxation processes is the coupling to a low-impedance electromagnetic environment. We can estimate the relaxation rate by $\Gamma_{in} \sim E(R_{env}/R_K)$ where $R_K = h/e^2$ is the quantum resistance and E is in the order of the Josephson energies $E \sim J_j, J_X$. Thus it is not difficult to satisfy the condition $\tau_{op} < \Gamma_{in}^{-1}$ experimentally. In fact, it has been found recently that inelastic relaxation times in charge qubits can be made quite large and exceed by far the typical dephasing times due to background charge fluctuations [18, 19].

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